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OPTIMAL PLASTIC DESIGN OF CIRCULAR PLATES

P. V. MARCAL

Imperial College of Science and Technology, London

Abstract—The method of optimal plastic design by an associated non-linearly elastic structure is used to find the minimum cost of circular plates with rotationally symmetric conditions of loading and support. A study is made of the built-in plate and an example is given where a minimum thickness of the plate is specified.

1. INTRODUCTION

THE optimal design of circular plates subjected to rotationally symmetric conditions of loading and support has been a subject of considerable interest in recent years. Hopkins and Prager [1] and Freiberger and Tekinalp [2] have investigated the case of a plate with uniformly distributed pressure and simply supported edge conditions.

Onat *et al.* [3] by applying the direct design procedure developed by Drucker and Shield [4] have investigated the case of a plate with arbitrarily distributed pressure and with both built-in and simply supported edge conditions.

In order to simplify the analysis, the design objective has usually been restricted to the minimization of the weight of the structure, the implicit assumption often being that the overall cost is some function of the amount of material used.

Although it relates to shells of constant thickness, it may be of interest to note here that Raymond and Osbeldiston [5] in arguing for the use of higher stresses in pressure vessel design have given some idea of the relation between weight and economic cost in the pressure vessel field. We quote "For a group of typical process vessels for chemical plant which were designed in I.C.I. to the German stress levels, the savings in both direct cost and weight are estimated to have been about 20 per cent. The savings to users of vessels extend beyond those on the direct costs, because the smaller weights reduce also the cost of foundations, supports and erection, and sometimes of cranes". It will be noticed that, even in this particular example, the overall savings to the user does not vary as a linear function of weight.

Another restriction found in the optimal design methods to date is the inability to impose minimum sections. Frequently a minimum section must be specified in the structure in order to carry the shear loads and in some cases to avoid buckling as well.

A general method of optimal design by means of an associated structure has recently been developed by Marcal and Prager [6]. It has the minimum cost of the structure as its design objective, the unit cost being a *convex* function of the plastic resistance. This design objective is more general than that of minimum weight and can include the latter as a particular case. Design is effected by means of an associated non-linearly elastic structure. This is supported and loaded in the same manner as the plastic structure that is to be designed, and the minimum of its complementary energy is made to correspond to the minimum of the cost of the plastic structure.

Statement of object

The object of the present paper is

(a) to develop, in greater detail, the application of the method of [6] to symmetrically loaded circular plates which obey the Tresca yield criterion;

(b) to study the case of a built-in plate subjected to uniformly distributed pressure; and

(c) to extend the method so as to allow for the inclusion of a minimum section, i.e. a minimum plastic resistance in the specific cost function of the design.

2. OPTIMAL PLASTIC DESIGN AND THE ASSOCIATED PLATE

In the present section, the method of optimal design developed in [6] is outlined in the form appropriate for a symmetrically loaded circular plate in order to make the present paper reasonably self-contained.

In the following, we use polar co-ordinates r, θ and take the centre of the plate as origin. The plate radius will be denoted by R.

Let $\phi(M_p)$ be the "cost" per unit area of building a plate that will support the fully plastic moment M_p, i.e. the bending moment, for cylindrical bending. $\phi(M_p)$ will be called the specific cost function; its graph will be assumed to be convex with respect to the M_p-axis.

The total cost Φ of the design is then given by

$$\Phi = 2\pi \int_0^R \phi(M_p) r \, \mathrm{d}r. \tag{2.1}$$

The object of an optimal design is to find, among all the designs capable of carrying the given loads, that design which also minimizes the total cost.

We begin by considering another plate for the same conditions of loading and support. The specific complementary energy c of this plate is taken to be equal to the specific cost function $\phi(M_p)$ of the original plate. This plate will be called the associated plate.

For the associated plate, the specific complementary energy is given by

$$c(M_p) = \int_0^{M_r} k_r \, \mathrm{d}M_r + \int_0^{M_\theta} k_\theta \, \mathrm{d}M_\theta = \phi(M_p) \tag{2.2}$$

where M_r, M_θ and k_r, k_θ are the radial and circumferential bending moments and curvatures of the plate.

Since

$$c(M_p) = \phi(M_p) = \int_0^{M_r} \frac{\mathrm{d}\phi}{\mathrm{d}M_p} \frac{\partial M_p}{\partial M_r} \mathrm{d}M_r + \int_0^{M_\theta} \frac{\mathrm{d}\phi}{\mathrm{d}M_p} \frac{\partial M_p}{\partial M_\theta} \mathrm{d}M_\theta \tag{2.3}$$

by (2.2), we have

$$k_r = \frac{\mathrm{d}\phi}{\mathrm{d}M_p} \frac{\partial M_p}{\partial M_r}, \qquad k_\theta = \frac{\mathrm{d}\phi}{\mathrm{d}M_p} \frac{\partial M_p}{\partial M_\theta} \tag{2.4}$$

wherever the partial derivatives $\partial M_p / \partial M_r$ and $\partial M_p / \partial M_\theta$ exist.

Solving the problem of the associated plate with the aid of (2.4), we obtain the strain field $= [k_r(r), k_\theta(r)]$ and the stress field $M^* = [M_r(r), M_\theta(r)]$.

With the above in mind, we return to the original plate and consider a design based on the statically admissible stress field M^*. Because of the definition of the associated plate, the field k^* can now be interpreted as a kinematically admissible strain rate field. The common factor $d\phi/dM_p$ in the two equations (2.4) represents the rate of increase of the specific cost with the plastic moment and is therefore positive, so that the strain rate field k^* is compatible with the stress field M^* in the sense of satisfying the normality flow rule of plasticity. Hence, by the uniqueness theorem of limit analysis, the design of the plate with the stress field M^* is at the plastic limit.

It will be shown that this design also minimizes the cost Φ.

For, by (2.2)

$$\Phi = 2\pi \int_0^R \phi(M^*) r \, dr$$
$$= 2\pi \int_0^R c(M^*) r \, dr \tag{2.5}$$

equals the complementary energy of the associated plate. But M^* is the stress field obtained from a solution of the associated plate so that the complementary energy and hence the cost Φ is a minimum. (Note that to establish the minimum character of the complementary energy, rather than merely its extremum character, we need the convexity of the specific complementary energy. This is the motivation for assuming the specific cost $\phi(M_p)$ to be a convex function of M_p.)

The design of the original structure with the stress field M^* is, therefore, an optimal one and the method of optimal design by an associated plate has been established.

3. DESCRIPTION OF PLATE

We consider a plate obeying Tresca's yield condition

$$M_p = \tfrac{1}{2}(|M_r| + |M_\theta| + |M_r - M_\theta|) \tag{3.1}$$

and deforming in accordance with the generalized flow rule.

In the present paper, attention is restricted to a built-in circular plate loaded over its whole surface by a uniformly distributed pressure of intensity p. The equilibrium equation of this symmetrically loaded plate is

$$(rM_r)' - M_\theta = -p\frac{r^2}{2} \tag{3.2}$$

where the prime denotes differentiation with respect to r.

The specific cost function $\phi(M_p)$ of the plate is assumed to be made up of two linear portions with slopes α_1 and $\alpha_1 + \alpha_2$ as shown in Fig. 1. The two lines meet at $M_p = Y$. In order that this function be convex to the M_p axis, $\alpha_2 \geq 0$. For convenience of discussion the fully plastic moment M_p has been defined as positive in (3.1). Strictly speaking M_p takes on a negative value when the moments are in the lower half of the Tresca hexagon (Fig. 3(a)) and, in order to show that the specific complementary energy and the specific cost function is always positive, the fully plastic moment M_p is allowed to appear as negative in Figs. 1 and 2.

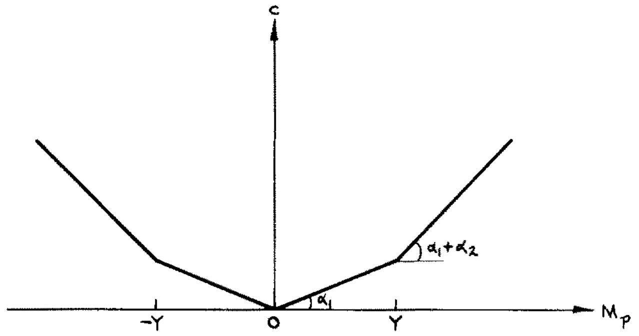

FIG. 1

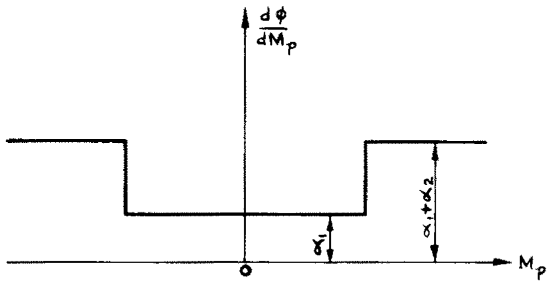

FIG. 2

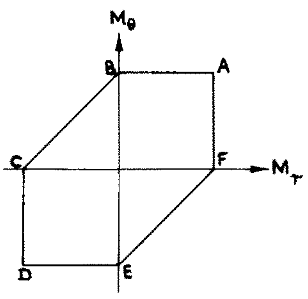

FIG. 3(a)

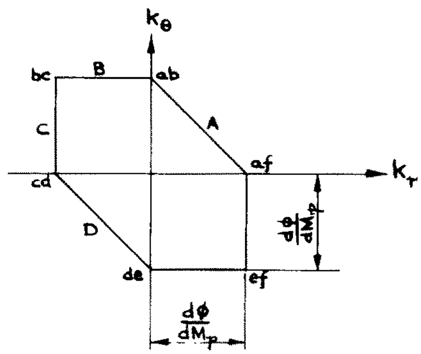

FIG. 3(b)

Note on specific cost functions

In [6] where a beam was considered, it was shown that the only restriction on the specific complementary energy of an associated structure was that the curve of the specific complementary energy versus the bending moment remained convex to the moment axis. This curve of the specific complementary energy was permitted to be piecewise differentiable. By the definition of the associated structure, this also held for the specific cost function of the original beam and hence permitting the use of piecewise linear specific cost functions.

By making the reasoning in [6] more general, it can be shown that the only restriction on the specific cost function of a plate is that its curve of specific cost function versus the plastic moment M_p remains convex to the plastic moment axis.

4. STRESS REGIMES FOR THE ASSOCIATED PLATE

This section considers the stress regimes on the hexagon representing the Tresca yield condition (Fig. 3(a)) in order to find the regimes which allow deformation of the associated plate without violating the geometric conditions of compatibility. The relations of compatibility are implicit in

$$k_r = -u'', \qquad k_\theta = -\frac{u}{r} \tag{4.1}$$

where u is the deflection of the horizontal plate and is taken to be positive when acting in a downwards direction.

The specific cost function in Fig. 1 has slopes $\mathrm{d}\phi/\mathrm{d}M_p$ which are shown plotted against the plastic moment in Fig. 2. When the plastic moment has the value Y, the derivative $\mathrm{d}\phi/\mathrm{d}M_p$ is considered as taking any value between α_1 and $\alpha_1+\alpha_2$ (see [6] for a continuity argument justifying this convention).

Assuming that the pressure on the horizontal plate only acts in the downward direction, we need only consider the stress regimes in the top half of the hexagon in Fig. 3(a). The curvatures of the deformed plate that can be caused by the bending moments on the hexagon of Fig. 3(a) are shown in Fig. 3(b).

Figure 3(b) is obtained in the following manner.

For the regime AF, $M_r = M_p$, $M_\theta = 0$ and by (2.4)

$$k_r = \frac{\mathrm{d}\phi}{\mathrm{d}M_p}\,.\,1, \qquad k_\theta = 0 \tag{4.2}$$

(4.2) is represented by af on Fig. 3(b).

For the regime A, we proceed in a manner which is analogous to that used in obtaining the strain rates by the generalized flow rule of plasticity.

At A the intersection of the two regimes $M_r = M_p$ and $M_\theta = M_p$.

$$M_p = \lambda M_r + (1-\lambda)M_\theta \tag{4.3}$$

and by (2.4)

$$k_r = \frac{\mathrm{d}\phi\lambda}{\mathrm{d}M_p}, \qquad k_\theta = \frac{\mathrm{d}\phi}{\mathrm{d}M_p}(1-\lambda) \tag{4.4}$$

where $1 \geq \lambda \geq 0$.

The curvatures that can be caused by moments in regime A are shown by the line marked A in Fig. 3(b).

Similarly Fig. 3(b) can be completed by considering the other regimes in turn.

On account of the compatibility condition (4.1), not all of the stress regimes allow deformation of the associated plate. Each stress regime must now be examined in turn to see which regimes allow deformation over an annular region of finite width.

The analysis is first carried out for a constant value of $\mathrm{d}\phi/\mathrm{d}M_p$. Let $\mathrm{d}\phi/\mathrm{d}M_p = \alpha$.

Stress regime AF

$$k_\theta = -\frac{u'}{r} = 0 \tag{4.5}$$

$$k_r = -u'' = \alpha. \tag{4.6}$$

Integrating (4.6) and dividing by r

$$-\frac{u'}{r} = \alpha + \frac{a_1}{r}. \tag{4.7}$$

The subscripted a is a constant of integration.
(4.5) and (4.7) are not compatible for variable r.
Regime AF does not allow deformation of the associated plate.
Regime A

$$k_r + k_\theta = -u'' - \frac{u'}{r} = \alpha \tag{4.8}$$

solving (4.8)

$$u = a_2 \log r - \tfrac{1}{4}\alpha r^2 + a_3 \tag{4.9}$$

$$u' = \frac{a_2}{r} - \tfrac{1}{2}\alpha r \tag{4.10}$$

$$k_r = \frac{a_2}{r^2} + \tfrac{1}{2}\alpha, \qquad k_\theta = -\frac{a_2}{r^2} + \tfrac{1}{2}\alpha. \tag{4.11}$$

Regime A allows deformation of the associated plate.
Regime AB

$$k_\theta = -\frac{u'}{r} = \alpha \tag{4.12}$$

$$k_r = -u'' = 0. \tag{4.13}$$

Integrating (4.13) and dividing by r

$$-\frac{u'}{r} = \frac{a_4}{r} \tag{4.14}$$

(4.12) and (4.14) are not compatible for variable r.
Regime AB does not allow deformation of the associated plate.

Similarly, it can be shown that regimes B and BC do not allow deformation of the associated plate, while regime C can be shown to allow deformation according to

$$-\frac{u'}{r} = -\alpha - \frac{a_5}{r} \geq 0. \tag{4.15}$$

We consider next the case when the plastic moment M_p is equal to Y and where $\mathrm{d}\phi/\mathrm{d}M_p$ can take any value between α_1 and α_2. The work is shortened considerably by noting that with a constant plastic moment M_p, regimes AB and BC are the only regimes which can satisfy a changing radial moment M_r as demanded by the equations of equilibrium (3.4).

For regime AB with variable $\mathrm{d}\phi/\mathrm{d}M_p$,

$$k_\theta = -\frac{u'}{r} = \frac{\mathrm{d}\phi}{\mathrm{d}M_p} \tag{4.16}$$

$$k_r = -u'' = 0 \tag{4.17}$$

solving (4.16) and (4.17)

$$\frac{d\phi}{dM_p} = \frac{a_6}{r} \tag{4.18}$$

and

$$u' = -a_6. \tag{4.19}$$

Regime AB allows a deformation of the associated plate.

Similarly, it can be shown that for regime BC with a variable $d\phi/dM_p$,

$$k_\theta = -\frac{u'}{r} = \frac{a_7}{r^2}. \tag{4.20}$$

5. STRESS RELATIONS FOR THE ASSOCIATED PLATE

In this section we integrate the equations of equilibrium (3.4) for the stress regimes which can support movement of the associated plate.

Regime A

$$M_r = M_\theta = M_p \tag{5.1}$$

$$(rM_p)' - M_p = pr^2 \tag{5.2}$$

$$M_r = M_p = -\frac{pr^2}{4} + C_1 \tag{5.3}$$

the subscripted C is a constant of integration.

Regime AB

$$M_\theta = M_p = Y \tag{5.4}$$

$$(rM_r)' - Y = -\frac{pr^2}{2} \tag{5.5}$$

$$M_r = Y - \frac{pr^2}{6} + \frac{C_2}{r}. \tag{5.6}$$

Similarly, *for regime* BC, $M_p = Y$

$$M_\theta - M_r = Y \tag{5.7}$$

$$M_r = Y \log r - \frac{pr^2}{4} + C_3 \tag{5.8}$$

and for regime C

$$M_r = -M_p, \qquad M_\theta = 0 \tag{5.9}$$

$$M_p = \frac{pr^2}{6} - \frac{C_4}{r}. \tag{5.10}$$

6. ANALYSIS OF A BUILT-IN PLATE

The equations obtained in the previous two sections can now be used for the analysis of the associated plate. A deformation pattern which is consistent with the boundary equations is first sketched. This deformation pattern will in general comprise several regions, the moments in each region being chosen to act at one of the stress regimes which have been found to allow deformation of the associated plate. The choice of the appropriate regime depends on the shape of the deformation pattern. In the following, we illustrate and amplify the method by considering the case of a built-in horizontal plate which is subjected to a uniformly distributed pressure. The pressure is assumed to act in a downward direction.

Initially, with a sufficiently low pressure, the plate can be designed using only the first portion of the specific cost function with slope α_1. The deformation of the plate takes the form shown in Fig. 4(a). The curvature of the deformed plate changes sign at some radius $r = \rho$. The circle $r = \rho$ divides the plate into two regions.

For region 1

$$R \geq r \geq \rho$$

u'' is positive, u' is negative which means that k_r is negative and k_θ is positive (4.1). The regime which will allow this deformation is the regime C.

Similarly for region 2

$$\rho \geq r \geq 0$$

k_r is positive and k_θ is negative and the moment is in regime A.

Having chosen the regimes for the regions 1 and 2, we are now in a position to consider the deformation of the plate in detail:
for region 1, by (4.15) and the boundary condition $u' = 0$ at $r = R$

$$u' = \alpha_1(r-R) \tag{6.1}$$

for region 2, by (4.10) and the boundary condition $u' = 0$ at $r = 0$

$$u' = -\tfrac{1}{2}\alpha_1 r. \tag{6.2}$$

At $r = \rho$, because of the continuity of slopes, we have

$$\alpha_1(\rho-R) = \tfrac{1}{2}\alpha_1\rho \tag{6.3}$$

$$\rho = \tfrac{2}{3}R. \tag{6.4}$$

The values that the curvatures take for each region of the plate are shown on the k_r vs. k_θ plane in Fig. 4(b).

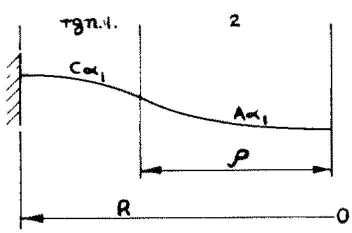

FIG. 4(a)

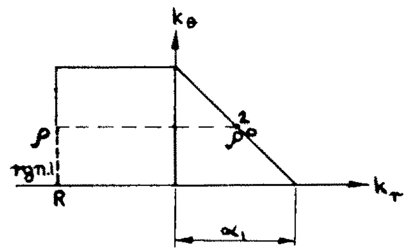

FIG. 4(b)

As we move along the radius of the plate, the path which these values take on the diagram can be followed in broad outline. Starting from the origin O and moving outwards, the curvatures first remain constant at the point marked region 2 until the radius ρ is reached. At the radius there is a discontinuous jump in the curvature k_r while the curvature k_θ remains constant because of the requirement of continuity of slopes. Both the points with constant k_θ are marked ρ in Fig. 4(b). Moving beyond the radius ρ, the curvature k_θ decreases until it becomes equal to zero at the radius R in accordance with the boundary condition of vanishing slope. Because this region is in regime C, the curvature k_r has a constant value of $-\alpha_1$. We call a path, such as the one described above, a curvature path.

Next, we consider the bending moments in the plate. For region 1, by (5.10) and the condition $M_r = 0$ at $r = \rho$ we have

$$M_r = -pr^2 + \frac{p\rho^3}{6r} = -M_p. \tag{6.5}$$

Substitute for ρ from (6.4)

$$M_p = -M_r = \frac{p}{6}\,\frac{r^3-(8/27)R^3}{r} \tag{6.6}$$

for region 2, by (5.3) and the condition $M_r = 0$ at $r = \rho$ we have

$$M_r = \frac{p}{4}(\rho^2 - r^2) = M_p. \tag{6.7}$$

Because our specific cost function is linear it can be interpreted as the specific weight of a sandwich plate so that, in the above, we have achieved the minimum weight design of a sandwich plate.

(6.4), (6.6) and (6.7) are the well known equations for the minimum weight design of a built-in sandwich plate obeying the Tresca yield criterion. (See for instance [3].)

As the pressure is increased a stage is reached when the plastic moment at radius R becomes equal to Y. The design for higher pressures using only the initial portion of the specific cost curve is no longer possible and the design with the above combination of regimes is no longer optimal. Both portions of the specific cost curve with slopes α_1 and $\alpha_1+\alpha_2$ must now be used.

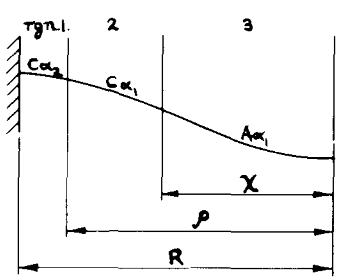

FIG. 5(a)

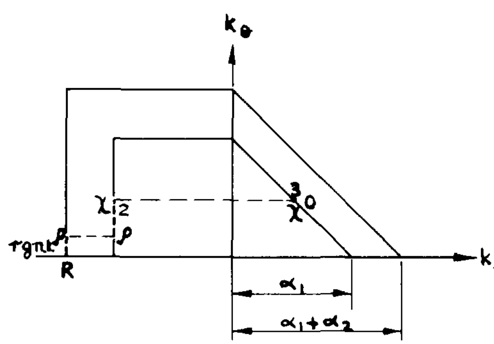

FIG. 5(b)

In order to achieve a smooth transition from the deformation pattern of the previous example, we take Fig. 5(a) as the next deformation pattern of the associated plate. There are now three regions and we denote the radii of their interfaces by ρ and χ respectively.

Using a similar analysis to the above, by inserting the geometric boundary conditions and equating slopes at $r = \chi$, we have

$$\chi = \tfrac{2}{3}[R(\alpha_1+\alpha_2)-\rho\alpha_2], \tag{6.8}$$

and by static considerations we have for regions 1 and 2

$$M_r = -\frac{pr^2}{6}+\frac{p\rho^3}{6r}-\frac{Y\rho}{r} = -M_p \tag{6.9}$$

$$\chi = \rho\left(1-\frac{6Y}{p\rho^2}\right)^{\frac{1}{2}} \tag{6.10}$$

(6.9) holds for regions 1 and 2 because the moments in both regions are at regime C. For region 3, we have

$$M_r = \frac{p}{r}(\chi^2-r^2) = M_p \tag{6.11}$$

This deformation pattern results in an optimal design providing the plastic moment at the centre of the plate does not exceed the value Y; by (6.11) we have

$$Y \geq \frac{p}{4}\chi^2. \tag{6.12}$$

With increase in pressure, the equality in (6.12) is soon reached. This can be seen from (6.6) and (6.7) of the previous deformation pattern which give

$$M_p = \frac{19}{18}\frac{pR^2}{9} \quad \text{at} \quad r = R$$

and

$$M_p = \frac{pR^2}{9} \quad \text{at} \quad r = 0. \tag{6.13}$$

Hence we see that this deformation pattern is of little practical interest, its only use is that it provides a link with the next deformation pattern which we now consider.

The deformation pattern is shown in Fig. 6(a) and the curvature path is traced in Fig. 6(b). There is now a new form of restriction on the validity of this deformation pattern (this was also found in [5]). This can best be appreciated by referring to Fig. 6(b). The curvature k_r at $r = \xi$ changes discontinuously while the curvature k_θ remains constant. At $r = \xi$ and with $(\alpha_1+\alpha_2)/2 > \alpha_1$, i.e. $\alpha_2 > \alpha_1$ the horizontal line from the point 2 on the outer hexagon of Fig. 6(b) can no longer meet the inner hexagon.

For the present we assume that $\alpha_2 \leq \alpha_1$ and proceeding as above, we have

$$\xi = \left[2\left\{(\rho-\tfrac{3}{2}\chi)\chi+(R-\rho)\chi\left(1+\frac{\alpha_2}{\alpha_1}\right)\right\}\right]^{\frac{1}{2}} \tag{6.14}$$

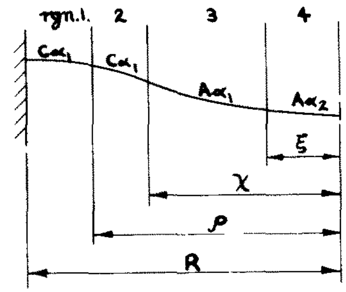

FIG. 6(a)

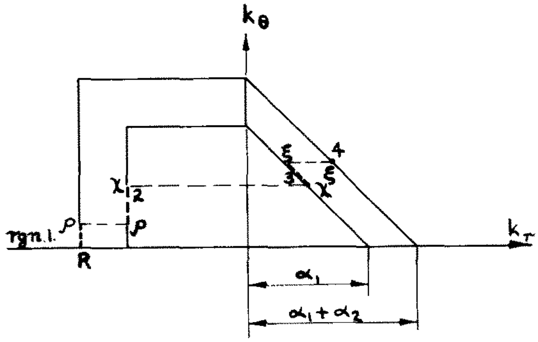

FIG. 6(b)

for regions 1 and 2,

$$M_r = -\frac{pr^2}{6}+\frac{p\rho^3}{6r}-\frac{Y\rho}{r} = -M_p, \tag{6.15}$$

$$\chi = \rho\left(1-\frac{6Y}{p\rho^2}\right)^{\frac{1}{3}}, \tag{6.16}$$

and for regions 3 and 4,

$$M_r = M_p = \frac{p}{4}(\chi^2-r^2), \tag{6.17}$$

$$\chi = \left(\frac{4Y}{p}+\xi^2\right)^{\frac{1}{2}}. \tag{6.18}$$

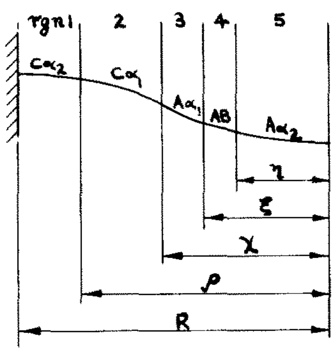

FIG. 7(a)

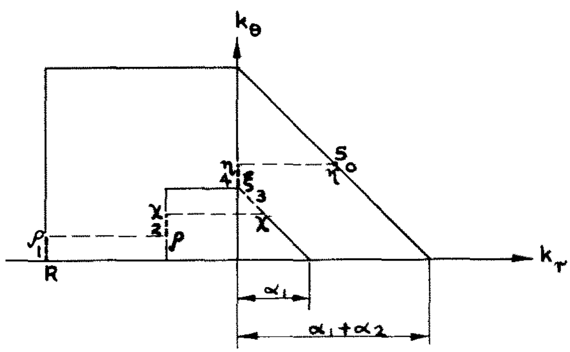

FIG. 7(b)

We now turn to the case when $\alpha_2 > \alpha_1$ and in order to join up the outer hexagon with the inner hexagon of Fig. 7(b) we have the deformation pattern shown in Fig. 7(a). A new region with moments in regime AB and constant plastic moment Y now appears. Again, by matching equations at the ends of the regions, we have

$$\xi = \left[2\left\{(\rho-\tfrac{3}{2}\chi)\chi+(R-\rho)\chi\left(1+\frac{\alpha_2}{\alpha_1}\right)\right\}\right]^{\frac{1}{2}} \tag{6.19}$$

$$\frac{\eta}{\xi} = \frac{2\alpha_1}{\alpha_1+\alpha_2} \tag{6.20}$$

for regions 1 and 2,

$$M_r = -\frac{pr^2}{6}+\frac{p\chi^3}{6r} = -M_p, \tag{6.21}$$

$$\chi = \rho\left(1-\frac{6Y}{p\rho^2}\right)^{\frac{1}{2}} \tag{6.22}$$

for region 3

$$M_r = \frac{p}{4}(\chi^2-r^2) = M_p, \tag{6.23}$$

for region 4

$$M_r = Y-\frac{pr^2}{6}+\frac{p\eta^3}{6r}, \qquad M_p = Y \tag{6.24}$$

$$\eta = \xi\left(\frac{3}{2}\frac{\chi^2}{\xi^2}-\frac{6Y}{p\xi^2}-\frac{1}{2}\right)^{\frac{1}{2}} \tag{6.25}$$

and for region 5,

$$M_r = Y+\frac{p}{4}(\eta^2-r^2) = M_p. \tag{6.26}$$

Region 3 of this deformation pattern decreases with increase of pressure. When region 3 becomes equal to zero we have the next and last deformation pattern shown in Fig. 8(a).

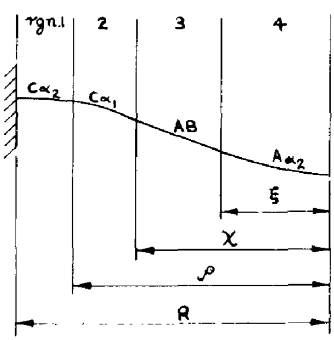

FIG. 8(a)

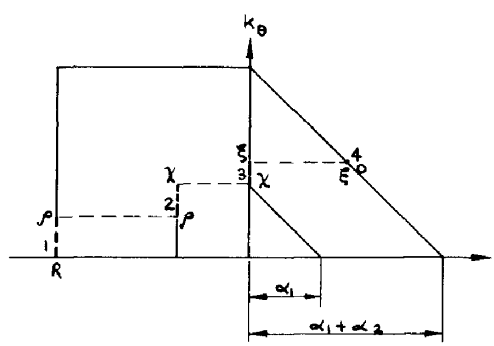

FIG. 8(b)

Proceeding as above, we have

$$\eta = R - \frac{\alpha_1}{\alpha_1 + \alpha_2}\chi - \alpha_1 \rho \tag{6.27}$$

for regions 1 and 2,

$$M_r = \frac{p}{6}\left(\frac{\chi^3}{r} - r^2\right) = -M_p \tag{6.28}$$

$$\chi = \rho\left(1 - \frac{6Y}{p\rho^2}\right)^{\frac{1}{3}} \tag{6.29}$$

for region 3, with $M_p = Y$

$$M_r = Y\left(1 - \frac{\chi}{r}\right) + \frac{p}{6}\left(\frac{\chi^3}{r} - r^2\right) \tag{6.30}$$

$$\eta = \chi\left(1 - \frac{6Y}{p\chi^2}\right)^{\frac{1}{2}} \tag{6.31}$$

and for region 4,

$$M_r = Y + \frac{p}{4}(\eta^2 - r^2). \tag{6.32}$$

The above equations were solved on a digital computer and the results are shown in Figs. 9 and 10. The r/R vs. pR^2/Y plot of Fig. 9 is for the case where $\alpha_2 > \alpha_1$. It shows the

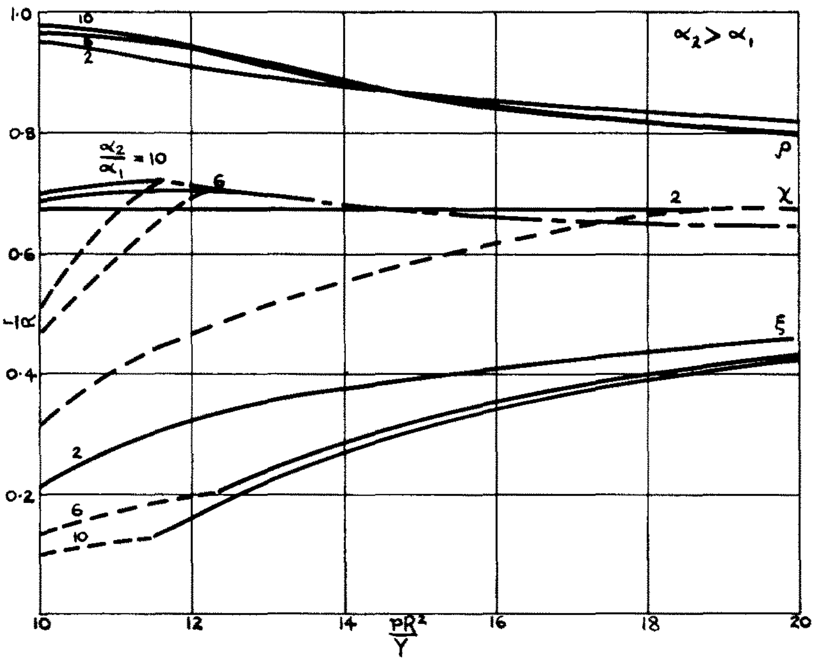

FIG. 9

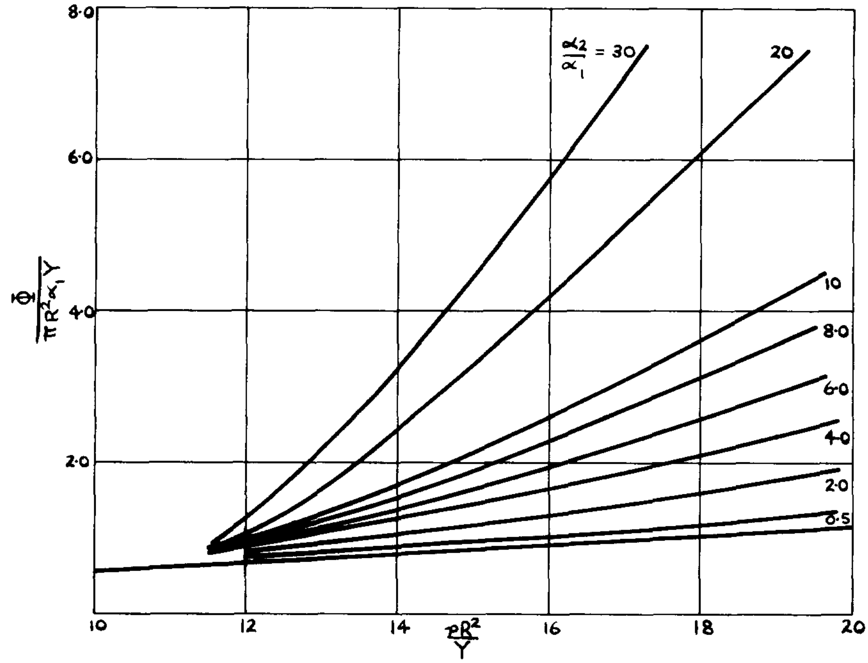

FIG. 10

variation in the size of the regions as the pressure is increased. The full and dashed lines show the results for the deformation pattern of Fig. 7(a). When two of the middle lines meet we have the deformation pattern of Fig. 8(a) and the results are shown by the broken lines.

Figure 10 shows the effect on the cost of increase in pressure for different ratios of α_2 to α_1.

Previously, optimal design of plates with a specified minimum section has not been possible. A design with a minimum cross section to carry the shear loads is obviously more realistic. As a final example of the present method, we consider the optimal design of a built-in plate with a specified minimum cross-section. The specific cost function for this case is obtained by letting $\alpha_1 = 0$ in Fig. 1. The cost of the minimum section is passive and does not enter into the analysis so that it can be added on after the analysis is completed. The deformation pattern for the associated plate is shown in Fig. 11(a). The change of curvature at $r = \chi$ requires that the radial moment at that point becomes equal to zero.

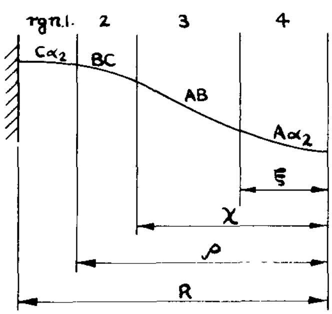

FIG. 11(a)

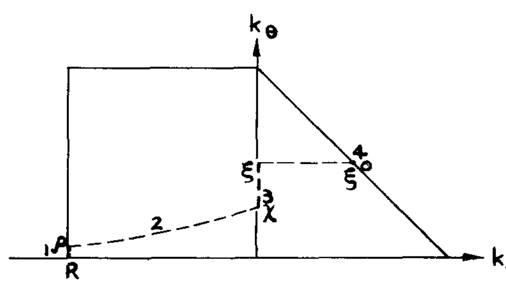

FIG. 11(b)

The only way we can achieve this and also satisfy the requirement of a minimum section is to let the moments in regions 2 and 3 operate in regimes BC and AB respectively, so that the regions 1 and 4 are now connected via two regions (2 and 3) with a constant section. Deformation of the associated plate in regions 2 and 3 takes place by virtue of the variable $d\phi/dM_p$ discussed in Section 4.

The curvature path corresponding to the deformation pattern of Fig. 11(a) and the stress regions discussed above is shown in Fig. 11(b).

To obtain the optimal design, we proceed as before and obtain

$$\xi = \frac{2(R-\rho)\rho}{\chi} \tag{6.33}$$

for region 1,

$$M_r = \frac{p}{6}\left(\frac{\rho^3}{r} - r^2\right) - \frac{Y\rho}{r} = -M_p \tag{6.34}$$

for region 2, $M_p = Y$

$$\frac{4Y}{p\chi^2}\left(1+\log\frac{\rho}{\chi}\right) + 1 - \frac{\rho^2}{\chi^2} = 0 \tag{6.35}$$

for region 3, $M_p = Y$

$$\xi = \chi\left(1 - \frac{6Y}{p\chi^2}\right)^{\frac{1}{2}} \tag{6.36}$$

and for region 4,

$$M_r = Y + \frac{p}{4}(\xi^2 - r^2). \tag{6.37}$$

Figure 12 shows the results for this example. The common points of the regions are shown in continuous lines. The dashed line gives the cost $\bar{\Phi}$ of the built-in plate. This cost $\bar{\Phi}$ is the cost over and above the cost of a plate with the minimum specified cross section.

SUGGESTIONS FOR FURTHER WORK

1. Optimal design is still in its infancy and, as far as the author is aware, no attempt has yet been made to check the validity of the theoretical results by experiment.
2. Though it is realized that the specific cost function is largely dependent on a particular application, some compilation of typical specific cost functions for particular cases may stimulate the application of the method to practical cases.
3. It would be interesting to see if the present methods could be extended to the optimal design of symmetrically loaded shells of revolution.

CONCLUSIONS

The method of optimal plastic design by means of an associated structure has been developed for symmetrically loaded circular plates which obey the Tresca yield criterion.

A design study was then made of the built-in plate subjected to uniform loading. The design study included the case where a minimum plastic resistance was specified in the specific cost function.

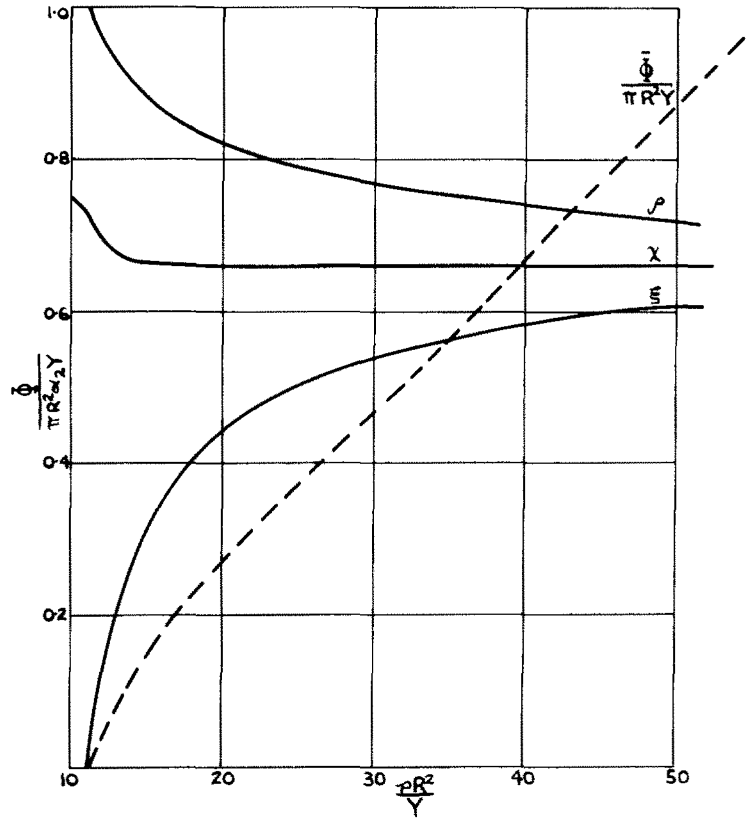

FIG. 12

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Résumé—La méthode de plan optimal plastique par une structure associée élastique non linéaire est employée afin de trouver le prix minimum d'épaisseur de plaques circulaires ayant des conditions rotationnellement symmétriques de chargement et de support. Une étude est faite de la plaque encastrée et un exemple est donné où une épaisseur minimum de la plaque est spécifiée.

Zusammenfassung—Die Methode des optimalen plastischen Entwurfes mittels einer assoziierten nichtlinearen elastischen Struktur wird angewandt um die minimalen Dickenkosten für Rundplatten mit drehungssymmetrischen Bedingungen für Belastung und Stützung zu ermitteln. Die eingebaute Platte wird gründlich untersucht und ein Beispiel wird gegeben wobei die Mindestdicke der Platte gegeben ist.

Абстракт—Применяется метод наилучшего пластического проекта при связанной нелинейной эластической структуре, чтобы найти стоимость минимальной тодщины циркулярных пластин со вращательно симметрическими условиями нагрузки и опоры. Произведено изучение вмонтированной пластины и даётся пример, где точно установлена минимальная толщниа пластины.